THE INTEGRAL EQUATION OF CERTAIN DYNAMIC CONTACT PROBLEMS OF ELASTICITY THEORY AND MATHEMATICAL PHYSICS

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Several dynamic mixed problems of elasticity theory, hydromechanics, and mathematical physics are reducible to the integral equations which are the subject of the present paper. The integral equations to be investigated are characterized by a kernel $k(t)$ which does not decrease as $t \to \infty$.

This fact makes it difficult to analyze the integral equations and excludes the direct application of the asymptotic expansion method developed in [1, 2] which is based on the use of the Wiener-Hopf equations.

We shall investigate a certain class of integral equations with the above properties, propose a method for constructing their solutions, and describe certain applications to dynamic contact problems.

1. Let us consider an integral equation of the form

$$\int_{-a}^{a} k(x-\xi)\, q(\xi)\, d\xi = \pi \cos \eta x, \quad |x| \leqslant a \tag{1.1}$$

$$k(t) = \int_{0}^{\infty} K(u) \cos ut\, du \tag{1.2}$$

Here $K(z)$ is an even function meromorphic in the complex plane which has no zeros or poles outside the coordinate axes and is real on the real axis.

We shall assume (see Remarks 4.1 and 4.2) that the asymptotic behavior of the imaginary zeros and poles of the upper half-plane is described by the relations

$$z_n \sim i(\beta n + b) + o(n^{-1}), \qquad \zeta_n \sim i(\beta n + g) + o(n^{-1}), \qquad n \to \infty$$

$$|\xi_n| < |z_n| \qquad 0 < b - g < \beta \tag{1.3}$$

The positive part of the real axis contains m zeros and p poles (which we denote by subscripts), i.e.

$$\operatorname{Im} z_n = 0, \quad n = 1, 2, \ldots, m; \qquad \operatorname{Im} \zeta_k = 0, \quad k = 1, 2, \ldots, p$$

We assume that there are no multiple zeros and poles. Clearly,

$$K(z) = A \prod_{k-1}^{\infty} \left(1 - \frac{z^2}{z_k^2}\right)\left(1 - \frac{z^2}{\zeta_k^2}\right)^{-1} \tag{1.4}$$

Let us introduce a function of the form

$$K_+(z) = \sqrt{A} \lim_{n\to\infty} \prod_{l=1}^{n} \left(1 + \frac{z}{z_k}\right)\left(1 + \frac{z}{\zeta_k}\right)^{-1} \tag{1.5}$$

which, as we can readily verify by recalling (1.3), exists and is meromorphic in the complex plane.

Lemma 1.1. The estimates

$$K(z) \sim c|z|^{-2\gamma} + o(z^{-2\gamma}), \quad |\arg z \pm \pi/2| > \varepsilon > 0$$
$$K_+(z) \sim c|z|^{-\gamma} + o(z^{-\gamma}), \quad |\arg z \to \pi/2| > \varepsilon > 0$$
$$K_+'(-z_l) \sim cl^{-\gamma} + o(l^{-\gamma}), \quad l \to \infty$$
$$[K_+^{-1}(-\zeta_l)]' \sim cl^{\gamma} + o(l^{\gamma}), \quad \gamma = (b-g)\beta^{-1}$$

are valid.

These estimates were derived by applying the Euler-Maclaurin summation formulas [3] to the logarithms of the functions $K(z)$, $K_+(z)$ taken in the sense of (1.5). Interpreting integral (1.2) as a principal value, we can express it as a series in residues of the form

$$k(t) = \sum_{k=1}^{p} is_k \sin\zeta_k|t| + \sum_{k=p+1}^{\infty} s_k \exp i\zeta_k|t|, \quad s_k = \frac{\pi i}{[K^{-1}(\zeta_k)]'} \tag{1.6}$$

It is clear from properties (1.3) and Lemma 1.1 that

$$k(t) = O(t^{2\gamma-1}), \quad \gamma \neq 0.5; \quad k(t) = O(\ln t), \quad \gamma = 0.5, \quad t \to 0 \tag{1.7}$$

and that series (1.6) converges uniformly for $t \geqslant \delta > 0$.

The kernel is merely bounded, rather than decreasing, at infinity (see Remark 4.1).

2. We shall attempt to find the solution of the integral equation in the form

$$q(\xi) = B_0 \cos\eta\xi + 2\sum_{l=1}^{\infty} x_l e^{iz_l a} \cos z_l \xi \tag{2.1}$$

Here $q(\xi) \in L_p(-a, a)$, $p < \gamma^{-1}$. It is clear that series (2.1) converges uniformly in any strictly interior interval to $[-a, a]$, if $\{x_l\} \in l_p, p > 1$, where $q(\xi)$ has singularities of order not exceeding $t^{-\gamma}$ at the points $-a$, a.

Bearing the above remarks in mind and comparing with (1.7), we conclude that it is possible to compute the integral on the left side of (1.1) by making use of expansions (1.6), (2.1). Making use of the results of [4], we arrive at the following system of linear algebraic equations equivalent to integral equations (1.1):

$$\sum_{l=1}^{\infty}\left(\frac{1}{\zeta_r - z_l} + \frac{\exp(-2ia\zeta_r) + \exp 2aiz_l}{\zeta_r + z_l} + \frac{\exp 2ai(z_l - \zeta_r)}{\zeta_r - z_l}\right)x_l =$$
$$= -2\frac{\exp - i\zeta_r a}{K(\eta)}\left[\frac{\cos(\zeta_r + \eta)a}{\zeta_r + \eta} + \frac{\cos(\zeta_r - \eta)a}{\zeta_r - \eta}\right] \quad (r = 1, 2, \ldots, p) \tag{2.2}$$

$$\sum_{l=1}^{\infty}\left(\frac{1}{\zeta_r - z_l} + \frac{\exp 2aiz_l}{\zeta_r + z_l}\right)x_l = 2\frac{\zeta_r \cos\eta a - i\eta\sin\eta a}{(\eta^2 - \zeta_r^2)K(\eta)} \quad (r = p+1, \ldots) \tag{2.3}$$

We note that the above system has the following properties.

1. Since the function $q(\xi)$ is real only if

$$\operatorname{Im} x_l e^{iz_l a} = 0 \qquad (l = 1, 2, \ldots, m)$$

it follows that the system is real.

2. If $m > 0$ or $p > 0$, then the coefficients of the infinite matrix have no limit as $a \to \infty$. This distinguishes system (2.2), (2.3) in an important way from the similar system obtained in [4,5]. This means, in turn, that if we have a solution of the equation on a semiaxis we cannot be certain that it is the limit of the solution of Eq. (1.1) as $a \to \infty$.

We conclude that the method of [1,2] does not enable us in this case to construct an approximate solution of Eq. (1.1).

In order to investigate system (2.2), (2.3) we rewrite it in the matrix form

$$[A + B(a)]X = D \tag{2.4}$$

$$A = \{a_{r,l}\} = \{(\zeta_r - z_l)^{-1}\}, \qquad X = \{x_l\} \in l_p, \quad p > 1$$

$$B(a) = \{b_{r,l}\} = \frac{\exp 2aiz_l + \exp(-2i\zeta_r a)}{\zeta_r + z_l} + \frac{\exp 2ai(z_l - \zeta_r)}{\zeta_r - z_l} \quad (r = 1, 2, \ldots, p)$$

$$B(a) = \{b_{r,l}\} = \frac{\exp 2aiz_l}{\zeta_r + z_l} \qquad (r = p + 1, \ldots) \tag{2.5}$$

Here $D = \{d_r\}$ is the sequence appearing in the right sides of Eqs. (2.2), (2.3).

3. The matrix $A + B(a)$ generates a certain operator which maps the elements of the space l_p, $p > 1$ into some space l_q. Let us investigate the properties of this matrix.

Theorem 3.1. The operator $B(a)$, $a > 0$ is completely continuous as an operator acting in l_p, $p > 1$.

It is sufficient to show that the order of the principal term of the sequence $B(a)X$, $(X \in l_p)$ as $r \to \infty$ is r^{-1}. This is easily accomplished.

We begin by showing that the operator A has the bounded inverse A^{-1} which acts in some l_q. Let us consider an integral of the form

$$J_n = \frac{1}{2\pi i} \frac{1}{K_+'(-z_l)} \int_{c_n} \frac{K_+(-t)\,dt}{(t-\eta)(z-t)}$$

Here c_n is a circle with its center at the origin and the radius R_n, $|z_n| < R_n < |\xi_{n+1}|$. If we make $n \to \infty$, then by virtue of the estimates of the Lemma, $J_n \to 0$. Using residue theory to compute the integral, we obtain a relation of the form

$$\sum_{r=1}^{\infty} \frac{1}{[K_+^{-1}(-\xi_r)]' K_+'(-z_l)(\zeta_r - \eta)(\zeta_r - z)} = \frac{1}{K_+'(-z_l)} \frac{K_+(-\eta) - K_+(-z)}{z - \eta}$$

Taking the limit of this expression as $z \to z_l$, $\eta \to z_k$, we obtain the equations

$$\sum_{r=1}^{\infty} \tau_{l,r} a_{r,k} = \begin{cases} 1 & (l = k) \\ 0 & (l \neq k) \end{cases} \tag{3.1}$$

Here

$$\tau_{l,r} = \frac{1}{[K_+^{-1}(-\zeta_r)]' K_+'(-z_l)(\zeta_r - z_l)} \tag{3.2}$$

Introducing the notation $A^{-1} = \{\tau_{l,r}\}$, we find that relation (3.1) implies that A^{-1} is the left inverse of the matrix A. In exactly the same way we can establish the fact that this matrix is also the right inverse of A.

Theorem 3.2. The matrix A^{-1} is the unique two-sided inverse of the matrix A (e.g. see [6]).

This theorem implies that the matrix A does not have a bilateral inverse other than that already constructed, for which the product $A^{-1}A\,A^{-1}$ is associative.

To prove this we must demonstrate the associativity of the product $A^{-1}AA^{-1}$, i.e. we must establish the absolute convergence of a double series. This is easy to do by applying the estimates of the lemma.

Theorem 3.3. If $d_r = 0^*(r^{-1})$ $(r \to \infty)$, then $A^{-1}D$ is the unique solution of the equation $AX = D$.

It is sufficient [6] to establish the associativity of the product $AA^{-1}D$.

Theorem 3.4. When acting in l_p, $p > (1-\gamma)^{-1}$ the operator $A^{-1}B$, is completely continuous.

By applying the estimates of the lemma to the elements of the sequence $A^{-1}BX$, $X \in l_q$, we obtain the relation

$$|y_r| = \|X\|\,O(r^{\gamma-1}) \qquad (r \to \infty)$$

which ensures the compactness of the set $Y = A^{-1}BX$ in l_p, $p > (1-\gamma)^{-1}$.

Theorems 3.1 - 3.4 enable us to represent Eq. (2.4) as a second-order equation with a completely continuous operator in l_p

$$X = -A^{-1}B(a)X + A^{-1}D \tag{3.3}$$

4. Let us consider the construction of the asymptotic solution of integral equation (1.1) for $a \to \infty$. As we see from (2.5), the elements of the matrix $B(z)$ are entire functions. Some of them vanish as $\operatorname{Re} z \to \infty$; the others are bounded.

Let us carry out the following matrix decomposition:

$$B(z) = P_m(z) + Q_p(z) + B_0(z) \tag{4.1}$$

The matrices on the right consist of the following elements:

$$P_m = \{p_{r,l}^{m}\}, \quad Q_p = \{q_{r,l}^{p}\}, \quad B_0 = \{b_{r,l}^{\bullet}\} \tag{4.2}$$

$$p_{r,l}^{m} = \begin{cases} (\zeta_r + z_l)^{-1}\exp 2ziz_l & (l = 1, \ldots, m;\ r = 1, 2, \ldots) \\ 0 & (l = m+1, \ldots) \end{cases}$$

$$q_{r,l}^{p} = \begin{cases} [(\zeta_r + z_l)^{-1} + (\zeta_r - z_l)^{-1}\exp 2ziz_l]\exp(-2zi\zeta_r) & (r = 1, \ldots, p) \\ 0, & (l = 1, 2, \ldots;\ r = p+1, \ldots) \end{cases}$$

$$b_{r,l}^{\bullet} = \begin{cases} (\zeta_r + z_l)^{-1}\exp 2ziz_l & (r = p+1, \ldots;\ l = m+1, \ldots) \\ 0 \end{cases}$$

It is obvious that all of the elements of the matrix $B_0(z)$ tend to zero as $\operatorname{Re} z \to \infty$. Thus, having constructed the solution of the infinite system

$$[A + P_m(z) + Q_p(z)]Y(m, p) = D \tag{4.3}$$

we arrive at the problem considered in [5].

The matrix A in Eq. (4.3) is perturbed by m infinite columns of the matrix P_m and by p infinite rows of the matrix Q_p. Solution of the problem of perturbation of an infinite matrix by a finite number of columns is considered in [5] where we construct the solution of the equation

$$A_1 Y(m, 0) \equiv [A + P_m] Y(m, 0) = D \tag{4.4}$$

It is therefore necessary to construct the solution of a system of the form

$$[A_1 + Q_p] Y(m, p) = D \tag{4.5}$$

The following lemma can be proved as in [5].

Lemma 4.1. The solution of the system

$$[A + Q_n(z)] T_n = D \tag{4.6}$$

is given by the recursion formula

$$T_n = \{t_l(n)\} = \left\{t_l(n-1) - \frac{\tau_{l,n}(n-1)\,\sigma(n)}{1 + \delta_{n,n}(n-1)}\right\}, \qquad T_0 = \{t_l(0)\}$$

Here

$$\begin{gathered}\sigma(n) = (Q_n - Q_{n-1}) T_{n-1}, \quad A^{-1} = \{\tau_{l,k}(0)\} \\ [A + Q_k]^{-1} = \{\tau_{l,m}(k)\} = \left\{\tau_{l,m}(k-1) - \frac{\tau_{l,k}(k-1)\,\delta_{k,m}(k-1)}{1 + \delta_{k,k}(k-1)}\right\} \\ \{\delta_{n,r}(k)\} = (Q_n - Q_{n-1})(A + Q_k)^{-1}\end{gathered} \tag{4.7}$$

The solution may not exist on a countable set of zeros of the function $1 + \delta_{n,n}(n-1)$. The lemma is valid if all the series appearing in (4.7) converge. This does, in fact, happen in our case.

Applying Lemma 4.1 to Eq. (4.5), we can rewrite initial equation (2.4) as

$$[A_2 + B_0(z)] X = D \tag{4.8}$$

The solution of this equation for $B_0 \equiv 0$ is clearly $Y(m, p)$. The matrix A_2^{-1} is also known. We can therefore apply the results of [4,5] to Eq. (4.8) and construct its solution in effective form throughout the entire right half-plane. This has the following significance. We can use the method of successive approximations to construct the solution of system (4.8) for sufficiently large $\operatorname{Re} z$. Expression (3.3) and the analyticity of the coefficients with respect to z imply that the resolvent has pole singularities only.

By the method of [5] we can continue the resolvent into the entire domain $\operatorname{Re} z > 0$. For $z = a$ we obtain the solution of system (2.4). Applying the method of [4], we conclude that the asymptotic properties of system (4.8) are described by the relation

$$X = Y(m, p) + O(\exp(-2a\,|z_{m+1}|)) \tag{4.9}$$

This means that it is necessary to use $Y(m, p)$ to construct the zeroth term of the asymptotic form of the solution of Eq. (1.1) for $a \to \infty$, and that the effectiveness of such a solution diminishes with decreasing $|z_{m+1}|$.

Remark 4.1. If it turns out that a pole of the function $K(z)$ is of multiplicity n, then a polynomial of degree $n - 1$ arises in front of the corresponding harmonic in

(1.6). A multiple pole on the real axis is interpreted as the limiting position of the poles sliding along the real axis. This is one of the methods of regularizing the integral.

Remark 4.2. If it turns out that a zero of the function $K(z)$ is of multiplicity n, then a polynomial of order $n-1$ arises in front of the corresponding harmonic in (2.1); if moreover, the zero is equal to η, then the degree of the above polynomial increases by unity.

Remark 4.3. The above properties of the kernel of integral equation (1.1) do not prevent us from using the methods of [7,8] for constructing the asymptotic solution for $a \to 0$.

5. Let us construct the zeroth term of the asymptotic form of the solution for $a \to \infty$ in the case where p and m vary from zero to unity.

1. We set $p = m = 0$. The zeroth term of the asymptotic form is given by the relation [6]

$$q(x) = \frac{\cos \eta x}{K(\eta)} - 2\sum_{l=1}^{\infty} \left[\frac{e^{i\eta a}}{(z_l+\eta)K_+(-\eta)} + \frac{e^{-i\eta a}}{(z_l-\eta)K_+(\eta)}\right] \frac{e^{iz_l a}\cos z_l x}{K_+'(-z_l)} + O(e^{-2a|z_1|}) \tag{5.1}$$

The problem of representing this series in effective form and of isolating the singularities is discussed in [1,2] and elsewhere, and need not be considered here.

2. We set $p = 1$, $m = 0$. Construction of the zeroth term of the asymptotic form clearly requires us to solve system (4.5) for $p = 1$, $m = 0$.

Let us first construct $Y(0, 0) = \{y_l(0)\} = A^{-1}D$. Making use of (2.2), (3.2), we obtain $Y(0, 0)$ in the form

$$y_l(0) = \left\{-\left[\frac{e^{i\eta a}}{(z_l+\eta)K_+(-\eta)} + \frac{e^{-i\eta a}}{(z_l-\eta)K_+(\eta)}\right]\frac{1}{K_+'(-z_l)} - \left[\frac{\zeta_1\cos\eta a - i\eta\sin\eta a}{\eta^2-\zeta_1^2} + \right.\right.$$
$$\left.\left. + \left(\frac{\cos(\zeta_1+\eta)a}{\zeta_1+\eta} + \frac{\cos(\zeta_1-\eta)a}{\zeta_1-\eta}\right)\exp -i\zeta_1 a\right]\frac{1}{K(\eta)\,[K_+^{-1}(-\zeta_1)]'\,K_+'(-z_l)(\zeta_1-z_l)}\right.$$

Now, making use of Formula (4.7), we compute $\delta_{1,1}(0)$, $\sigma(1)$. This yields relations of the form

$$\delta_{1,1}(0) = \frac{e^{-2ai\zeta_1}}{[K_+^{-1}(-\zeta_1)]'}\left[\frac{1}{2\zeta_1 K_+(\zeta_1)} - \frac{1}{2\pi i}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{e^{-2aiz}\,dz}{K_+(z)(\zeta_1+z)^2}\right]$$

$$\sigma(1) = \frac{e^{-2ai\zeta_1}}{K(\eta)}\left\{\frac{1}{2\pi^2}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{(t\cos\eta a - i\eta\sin\eta a)K_+(-t)}{\eta^2-t^2}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{e^{-2aiz}\,dz\,dt}{K_+(z)(z+\zeta_1)(z+t)} + \right.$$
$$+ \frac{1}{\pi i K_+(\zeta_1)}\int_{-\infty+i\varepsilon}^{\infty+i\varepsilon}\frac{(z\cos\eta a - i\eta\sin\eta a)K_+(-z)\,dz}{(\eta^2-z^2)(\zeta_1+z)} + \left[-\frac{1}{\zeta_1 K_+(\zeta_1)} + \right.$$
$$\left.\left. + \frac{1}{\pi i}\int_{-\infty-i\varepsilon}^{\infty-i\varepsilon}\frac{e^{-2aiz}\,dz}{K_+(z)(\zeta_1+z)^2}\right]\left[\frac{\cos(\zeta_1+\eta)a}{\zeta_1+\eta} + \frac{\cos(\zeta_1-\eta)a}{\zeta_1-\eta}\right]\frac{e^{-ia\zeta_1}}{[K_+^{-1}(-\zeta_1)]'}\right\}$$
$$(0 < \varepsilon < |z_1|)$$

Next, applying Lemma 4.2, we find that $y_l(1)$ is given by

$$y_l(1) = y_l(0) - \frac{\tau_{l,1}(0)\,\sigma(1)}{1 + \delta_{1,1}(0)}$$

Thus, the zeroth term of the asymptotic form of the solution of the integral equation becomes

$$q(x) = \frac{\cos \eta x}{K(\eta)} + 2 \sum_{l=1}^{\infty} y_l(1)\, e^{iz_l a} \cos z_l a + O(e^{-2a|z_1|}) \quad (a \to \infty) \tag{5.2}$$

3. We set $p = 0$, $m = 1$. To construct the zeroth term of the asymptotic form of the solution of the integral equation we need merely solve Eq. (4.4) for $m = 1$. We can do this with the aid of a lemma of [5].

The solution in this case can be written as

$$y_l(1) = y_l(0) - y_1(0) \frac{\varepsilon_{l,1}(0)}{1 + \varepsilon_{1,1}(0)}, \quad \varepsilon_{l,1}(0) = \frac{\exp 2aiz_1 K_+(z_1)}{K_+'(-z_l)(z_1 + z_l)} \tag{5.3}$$

The value of $y_l(0)$ is given by relation (5.1).

The zeroth term of the asymptotic form of the solution can be expressed in the form (5.2), where $y_l(1)$ is given by (5.3), and where the remainder term is of the order $0(\exp(-2a|z_2|))$.

4. We set $p = 1$, $m = 1$. In this case, applying the results of Sect. 2, we can construct y_l (1.1) according to the formula of the case 3.

The order of the remainder term is $O(\exp(-2a|z_2|))$.

6. Let us consider the problem of torsion of an elastic layer of thickness h by a die of width $2b$. We assume that the die vibrates harmonically in such a way that the displacements of the points directly under the die face are given by the relations

$$W^\circ(x, y, t) = \operatorname{Re} w(x, y)\, e^{-i\omega t} = \cos\eta x \cos \omega t, \qquad y = h, \qquad |x| \leqslant b \tag{6.1}$$

The layer is either (a) rigidly attached to a nondeformable base, or (b) rests on a nondeformable base without friction.

We are required to determine the contact stresses under the die in the case of steady vibrations.

This problem is reducible to the solution of the following boundary value problem for the Helmholtz equation:

$$\begin{gathered}
\Delta w + \varkappa^2 w = 0, \ |x| < \infty, \ 0 \leqslant y \leqslant 1 \\
\varkappa^2 = \rho\omega^2 h^2 G^{-1}, \quad a = b/h \\
w = \cos \eta x, \ |x| \leqslant a, \quad \partial w/\partial y = 0, \quad |x| > a, \quad y = 1 \\
\text{(a)} \qquad w = 0, \quad |x| < \infty, \quad y = 0 \\
\text{(b)} \quad \partial w/\partial y = 0, \quad |x| < \infty, \quad y = 0
\end{gathered} \tag{6.2}$$

where ρ and G are the density and shear modulus of the layer material.

It is well known, however, that such a problem is not correctly formulated until the conditions at infinity have been specified.

These conditions will be given in the course of deriving the integral equation of the problem.

We can derive the integral equation of the problem in terms of the contact stresses by Fourier transformation. To this end we solve an ancillary boundary value problem which is described by equation and boundary conditions (6.2) everywhere except on the boundary $y = 1$, where we have the condition

$$\partial w/\partial y = q(x), \qquad |x| < \infty, \qquad y = 1 \tag{6.3}$$

It is now easy to verify that the solution of both problems in the Fourier transforms $W(\alpha, y)$ is of the form

$$\text{(a)} \quad W(\alpha, y) = \frac{\operatorname{sh} \sigma y}{\sigma \operatorname{ch} \sigma} Q(\alpha) = W_0(\alpha, y) Q(\alpha) \tag{6.4}$$

$$\text{(b)} \quad W(\alpha, y) = \frac{\operatorname{ch} \sigma y}{\sigma \operatorname{sh} \sigma} Q(\alpha) = W_0(\alpha, y) Q(\alpha) \tag{6.5}$$

$$\sigma = \sqrt{\alpha^2 - \varkappa^2}$$

Next, returning to the function $w(x, y)$ by means of the formula

$$w(x, y) = \frac{1}{2\pi} \int_{\Gamma} W(\alpha, y) e^{-i\alpha x} d\alpha \tag{6.6}$$

we choose the integration contour Γ in such a way as to ensure fulfillment of the required conditions at infinity.

It is clear that for sufficiently large $\varkappa$ in the case of problem (a) and for all $\varkappa$ in the case of problem (b) the function $W_0(\alpha, y)$ has a finite number p of poles on the real axis which lie symmetrically with respect to the origin. The function $W_0(\alpha, y)$ is clearly the Fourier transform of the solution $w_0(x, y)$ in the case of a "point source" (a concentrated force).

The contour in (6.6) must be chosen in accordance with the required behavior of the function $w_0(x,y)$ as $|x| \to \infty$.

Let us consider some examples.

1. The contour Γ coincides with the real axis and integral (6.6) is interpreted in the sense of a principal value.

In this case the function $w_0(x, y)$ for problems (a) and (b) can be expressed in the form

$$\text{(a)} \quad w_0(x, y) = 0.5 \sum_{k=1}^{p} a_k \operatorname{sh} \sigma_k y \sin \zeta_k |x| + \sum_{k=p+1}^{\infty} a_k \operatorname{sh} \sigma_k y e^{i\zeta_k |x|} \tag{6.7}$$

$$\text{(b)} \quad w_0(x, y) = 0.5 \sum_{k=1}^{p} b_k \operatorname{ch} \sigma_k y \sin \zeta_k |x| + \sum_{k=p+1}^{\infty} b_k \operatorname{ch} \sigma_k y e^{i\zeta_k |x|} \tag{6.8}$$

$$\sigma_k = \sqrt{\zeta_k^2 - \varkappa^2}$$

Here a_k, b_k are certain constants and ζ_k are the poles of the functions $W_0(\alpha, y)$ corresponding to specific problems (see below); the first p of these poles are real.

Let us multiply relations (6.7), (6.8) by the time factor $e^{-i\omega t}$. It is then clear that "departing waves" of the form

$$\exp [i(\zeta_k |x| - \omega t)] \qquad (k \leqslant p) \tag{6.9}$$

are not absorbed at infinity, but are reflected and return in the form of "arriving waves" of the form

$$\exp [-i(\zeta_k |x| + \omega t)] \tag{6.10}$$

This phenomenon can be interpreted in a different way, i.e we can imagine that there is a source at infinity which generates arriving waves of the form (6.10).

2. The contour Γ curves above the negative poles and below the positive poles, intersecting the real axis at the origin. In this case the function $w_0(x, y)$ can be expressed in the form

$$\text{(a)}\quad w_0(x, y) = \sum_{k=1}^{\infty} a_k \operatorname{sh} \sigma_k y e^{i\zeta_k |x|}, \qquad \text{(b)}\quad w_0(x, y) = \sum_{k=1}^{\infty} b_k \operatorname{ch} \sigma_k y e^{i\zeta_k |x|} \qquad (6.11)$$

Departing waves of the form (6.9) are in this case absorbed at infinity, and waves of the form (6.10) do not arise.

It is easy to see that in Example 1 the points of the layer oscillate in the same phase as the stimulating force for $y = 1$ for all $\varkappa$, while Example 2 is characterized by a phase shift.

Choosing the appropriate integration contour Γ in relation (6.6) and introducing (6.4) or (6.5) for $y = 1$ on the basis of (6.2), we obtain the required integral equation in the form (1.1).

From now on we shall choose the contour Γ as in Example 1. The function $K(\alpha)$ then assumes the following values:

$$\text{(a)}\quad K(\alpha) = \sigma^{-1} \operatorname{th}\sigma, \qquad \text{(b)}\quad K(\alpha) = \sigma^{-1} \operatorname{cth}\sigma$$

The distributions of zeros and poles of the function $K(\alpha)$ in the upper half-plane are given by

$$\text{(a)}\quad z_n = i\sqrt{n^2\pi^2 - \varkappa^2} \sim in\pi + O(n^{-2}),$$

$$\zeta_n = i\sqrt{(n-0.5)^2\pi^2 - \varkappa^2} \sim i\pi(n-0.5) + O(n^{-2})$$

$$\text{(b)}\quad z_n = i\sqrt{(n-0.5)^2\pi^2 - \varkappa^2} \sim i\pi(n-0.5) + O(n^{-2})$$

$$\zeta_1 = \varkappa, \quad \zeta_n = i\sqrt{n^2\pi^2 - \varkappa^2} \sim in\pi + O(n^{-2}). \qquad n \geqslant 2$$

The above relations show that in the case of Problem (a) for small $\varkappa$ all the zeros and poles of the function $K(\alpha)$ lie on the imaginary axis. As $\varkappa$ increases they begin to sink towards the real axis (with their order preserved). The poles reach the real axis first. Combining to form a double pole at zero, they then move away from the origin in opposite directions along the real axis. The zeros behave in the same way with increasing $\varkappa$. For $\varkappa = 0$ we have an ordinary statics problem. In Case (b) the initial problem is not correctly posed for $\varkappa = 0$. On the other hand, if $\varkappa > 0$, then there are two poles, one on each side of the origin, lying on the real axis. The zeros, followed by the poles, sink towards the real axis with increasing $\varkappa$, etc.

The problem of torsion of an infinite elastic shaft of radius h by a die whose base shape varies according to the above law also reduces to integral equation (1.1). The function $K(\alpha)$ in this case is of the form

$$K(\alpha) = \frac{I_1(\sigma)}{\sigma I_0(\sigma) - 2I_1(\sigma)}, \qquad \sigma = \sqrt{\alpha^2 - \varkappa^2}$$

where $I_k(\sigma)$ is a Bessel function of an imaginary argument.

The problem of vibration of a plate on the surface of an ideal fluid layer (in its ideal formulation) also reduces to the above integration equation. In this case

$$K(\alpha) = \frac{\sigma \operatorname{sh}\sigma}{\nu \operatorname{ch}\sigma - \sigma \operatorname{sh}\sigma}, \qquad \nu = \frac{\omega^2 h}{g}$$

The same class of integral equations subsumes mixed problems on the compression and bending of an elastic strip by a vibrating die.

Remark 6.1. If the contour Γ in relation (6.6) has been chosen as in Example 2, then the corresponding infinite system is of exactly the same form as system (2.19) of [4]. This system can be investigated by the method proposed in [5].

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ASYMPTOTIC SOLUTION OF THE CONTACT PROBLEM FOR A THIN ELASTIC LAYER

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The contact problem of impressing a stamp in an elastic layer of finite thickness h lying without friction or adhering rigidly to an undeformable foundation is considered. The frictional forces between the stamp and the surface layer are assumed absent, and the surface layer outside the stamp is not loaded. The contact domain Ω between the stamp and the layer is assumed simply connected (*) and fixed.

An asymptotic solution of the above-mentioned problem has been obtained in [1-3] under the assumption that the relative thickness of the layer is sufficiently large, i.e. the dimensionless parameter $\lambda = h / a$, $a = {}^1/_2 \max R_{PQ}$ for any P and $Q \in \Omega$, is large.

A scheme for constructing the asymptotic solution of the mentioned problem under the assumption that the relative thickness of the layer is small has been expounded in [4].

*) Simple connectedness is assumed just for simplicity. The asymptotic method expounded below for the solution can be utilized even in the case of a multiply connected domain Ω.